Orbits can't favor hemispheres

Vaughan Pratt
Stanford University

October 21, 2014

Abstract

The quantitative aspects of Milankovitch theory have been determined numerically, greatly facilitated by the advent of computers. In this note we draw attention to a simple analytic proof of a basic aspect of the theory, namely that annual top-of-atmosphere insolation is apportioned equally between the hemispheres.

The basis of Milankovitch theory is that the obliquity or tilt ε of Earth's spin axis relative to its orbital axis distributes each hemisphere's annual insolation further polewards with larger tilt. Yet one can find "Earth is now closest to the sun in January ... thus favoring growth of glaciers and ice caps in the Northern Hemisphere" [1] in the literature when the fact is that each hemisphere receives the same total annual top-of-atmosphere insolation. By way of drawing attention to the correct account we give a short proof of this fact using elementary calculus and Kepler's second law of planetary motion.

Figure 1 depicts Earth's orbit about the Sun, $\odot$, with its orbital axis perpendicular to the page. Earth, $\oplus$, moves counterclockwise around the orbit at increasing longitude $\varphi = \angle V\odot\oplus$. While Earth's distance r from the Sun and angular velocity $\omega = d\varphi/dt$ are functions $r(t)$ and $\omega(t)$ of time, by Kepler's second law of planetary motion $\omega(t)r^2(t)$ is some constant, call it $K_\oplus$.

The remaining six points on the orbit are the vernal and autumnal equinoxes V and A at longitudes 0° and 180°, the summer and winter solstices S and W at 90° and 270°, perihelion P at longitude ϖ as the closest point of the orbit to the Sun, and aphelion P' as the furthest point. The center C of the ellipse is also the midpoint of PP', archaically the line of aspides. The eccentricity e of the orbit is the quotient $|C\odot|/|CP|$, namely the Sun's distance from the center C as a fraction of the way to perihelion P.

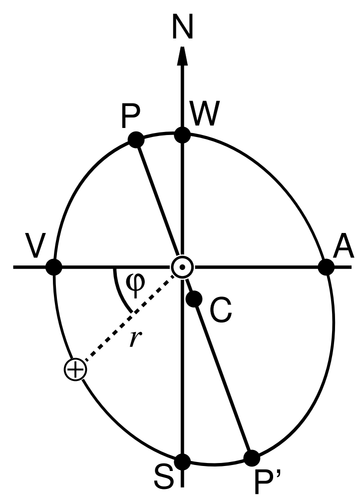

Figure 1 Earth's orbit (eccentricity exaggerated)

Earth's rotational or spin axis is tilted northwards (N) from its orbital axis by an angle ε, the axial tilt or obliquity, currently $23.44°$. Viewed from the Sun, Earth appears as a disk, but if we treat Earth as transparent its equator appears as an ellipse with semimajor axis equal to Earth's radius. Were the obliquity $90°$, the equator would appear at either equinox as a straight line of zero area, at either solstice as a circle the same apparent area as Earth, and at longitude φ as an ellipse of area a fraction $\sin(\varphi)$ of Earth's apparent area. With obliquity ε this fraction drops to $\sin(\varphi)\sin(\varepsilon)$, hence $\pm\sin(\varepsilon)$ at the solstices.

Let N and S denote the insolation in watts received by the corresponding hemispheres. When the equator as seen from the Sun appears as an ellipse favoring the Northern Hemisphere, N's share increases by half an ellipse while S's decreases by the same amount, so $N - S$ corresponds to the whole ellipse. Hence the northern surplus as a fraction of the total $N+S$, namely $(N-S)/(N+S)$, is simply another way of expressing the foregoing fraction $\sin(\varphi)\sin(\varepsilon)$.

At the solstices S and W, $\sin(\varphi)$ is respectively 1 and -1. The northern surplus is therefore actually a deficit at W: winter is cold. This would be true even for a circular orbit, $e = 0$, but for larger e we must take into account that N and S depend not only on φ and ε but also on distance r from the Sun, in inverse proportion to r^2. The northern surplus in watts at time t in years is therefore $k\sin(\varphi(t))/r^2(t)$ for some constant k, treating $\sin(\varepsilon)$ as part of k.

Figure 1 shows perihelion as near the winter solstice, about $13°$ east (counterclockwise) of it. From this it is easily seen that the northern surplus will be less positive in the northern summer than negative in the northern winter. Furthermore Earth is further from the Sun at aphelion than at perihelion in proportion to $(1+e)/(1-e)$. When e is 5%, as it has been at times in the past million years, the square of this proportion exceeds 22%!

But we should not immediately conclude that the Southern Hemisphere is the net beneficiary of a year's insolation. By Kepler's second law, Earth races through perihelion but dawdles at aphelion. The considerable heat the Northern Hemisphere loses to the Southern in any one second or day of its winter might therefore be made up, more or less, by having a longer albeit cooler summer.

The question then becomes, more or less? But this is just the sign of

$$\int_{t_0}^{t_0+1} \frac{\sin(\varphi(t))}{r^2(t)} dt.$$

The indefinite integral must be some constant times $-\cos(\varphi)$, as can be seen by taking the latter's derivative with respect to time t. By the chain rule this is $\sin(\varphi)d\varphi/dt$. But by Kepler's second law, $d\varphi/dt$ is $K_\oplus/r^2$. Hence the derivative of $-\cos(\varphi)$ is $K_\oplus \sin(\varphi)/r^2$, so the integral is $-\cos(\varphi)/K_\oplus$. Since $\cos(\varphi(t_0+1)) = \cos(\varphi(t_0))$ for any starting time t_0 the definite integral vanishes.

We conclude that orbital elements cannot favor one hemisphere of a planet over the other regardless of their values. Any hemispheric asymmetry in heating must therefore be accounted for by asymmetries in albedo, land-sea ratio, etc.

References

[1] Hansen, J.E., and M. Sato, 2012. In *Climate Change: Inferences from Paleoclimate and Regional Aspects*, Eds. A. Berger et al, Springer, 21-48.